A General Averaging Theory via Series Expansions

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Abstract This paper generalizes averaging theory to arbitrary order by synthesizing series expansion methods for nonlinear time-varying vector fields and their flows with nonlinear Floquet theory. A companion paper uses these results to construct exponentially stabilizing controllers for underactuated nonlinear systems.

1 Introduction

The method of averaging provides a useful means to study the behavior of non-linear dynamical systems under periodic forcing. Such studies are useful in the control of underactuated nonlinear systems. This paper provides a more coherent structure to the theory of averaging. Our approach consolidates the classical results of Sanders and Verhulst [1], and Bogoliubov and Mitropolsky [2]. More importantly, we extend averaging methods to arbitrary order of approximation. As an example, we give explicit calculations for 3^{rd} and 4^{th}-order averaging, and a general algorithm for calculating higher order averages. A companion paper uses these results to construct exponentially stabilizing feedback controllers for underactuated nonlinear systems [3].

Our work unites ideas from the areas of series expansions and averaging in the framework of nonlinear Floquet theory. The series expansions used here have their roots in the work of Magnus [4] and Chen [5]. Subsequently, Agrač̌hev and Gamkrelidze provided a better means to understand the series' convergence and to formulate expansions for systematic computation [6]. Agrač̌hev, Gamkrelidze, and Sarychev [7] have sought to better understand nonlinear control by using the series expansions. Susmann and Kaswki [8, 9, 10] have studied the mathematical structure underlying series expansions and its connection to controllability.

We wish to further this line of work and obtain a foundation for the use of series expansions and averaging for nonlinear control. In a similar vein, Bullo [11] developed a series expansion for *simple mechanical systems* under time-periodic kinematic motions. Martínez and Cortés [12] extend Bullo's results to arbitrary periodic motions.

By appealing to nonlinear Floquet theory, our work shows how series expansions fit within the method of averaging, and how they may be used to obtain arbitrary orders of approximation. Sarychev [13, 14] has considered related problems in the area vibrational control, giving means to use series expansions and nonlinear Floquet theory as a foundation for averaging.

Averaging is a classical theory. Sanders and Verhulst [1] give a comprehensive treatment. Note that they provide a formula for the average of a time-periodic vector field up to second order, and theorems relating the stability of the flow of time-periodic vector fields and their averaged vector field. Bogoliubov and Mitropolsky [2] do likewise, however they also give a general algorithm for the calculation of higher order averages. Guckenheimer and Holmes [15] use the Poincaré map to prove stability of the original flow via an analysis of the averaged flow. Although restricted to lower order averaging, it is similar to the technique of Sarychev.

Section 2 reviews relevant results from classical averaging theory that are subsequently generalized. Section 3 summarizes the representation of flows and vector fields by series expansions. These expansions are key to the development of our generalized averaging theory via nonlinear Floquet theory in Section 4. Section 5 discusses the calculation of higher order averages.

2 Classical Averaging Theory

Here we review the key ideas that will be subsequently generalized. The standard form of the equations of motion for averaging are,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \epsilon X(x,t), \quad x(0) = x_0, \tag{1}$$

where X is T-periodic, i.e., $X(x,t) = X(x, t+T)$. The average of X is typically given as,

$$\overline{X(x,t)} = \frac{1}{T}\int_t^{t+T} X(x,\tau)\,\mathrm{d}\tau\,, \tag{2}$$

where x is considered fixed. Often, $\overline{X(\cdot,t)}$ will be written as $\overline{X}(\cdot)$[1]. The average defines new autonomous equations,

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \epsilon\overline{X}(z), \quad z(0) = x_0. \tag{3}$$

Averaging theory seeks to determine conditions and degree of coincidence between the flows of (1) and (3), and the stability relations of these flows.

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[1]Via a change of coordinates, the average can instead be written: $\overline{X(x,t)} = \int_0^T X(x,\theta)\,\mathrm{d}\theta$.

Theorem 1 (first order averaging) *[1] Consider the initial value problems (1) and (3) with $x, z, x_0 \in M \subset \mathbb{R}^n$, $t \in [t_0, \infty)$, $\epsilon \in (0, \epsilon_0]$. Suppose that the following all hold: (1) $X(x,t)$ is Lipschitz-continuous in x on M, $t \geq 0$, continuous in x and t on $M \times \mathbb{R}^+$, and (2) $y(t)$ belongs to an interior subset of M on the time scale $\frac{1}{\epsilon}$. Then, $x(t) - y(t) = O(\epsilon)$, as $\epsilon \downarrow 0$ on the time scale $\frac{1}{\epsilon}$.*

Additionally let the following conditions be met: (1) $y = 0$ is an asymptotically stable fixed point, and (2) $\overline{X}$ is continuously differentiable in M, and has a domain of attraction $M^* \subset M$. If $x_0 \in M^*$,then, $x(t) - y(t) = O(\delta(\epsilon))$, for $0 \leq t < \infty$, with $\delta(\epsilon) = o(1)$.

An asymptotically stable fixed point in the average renders the approximation valid for all time.

Theorem 2 *[15] Suppose instead that $X(x,t)$ is C^r, $r \geq 2$, and bounded on bounded sets. If z^* is a hyperbolic fixed point of (3), then there exists an ϵ_0 such that for all $\epsilon \in (0, \epsilon_0]$, (1) possesses a unique hyperbolic periodic orbit $\gamma_\epsilon(t) = z^* + O(\epsilon)$ of the same stability type as z^*.*

The proof is based on the use of the average as a Poincaré map of the actual flow. The theorem does not preclude the case where the orbit is degenerate.

Theorem 3 *[16] Assume that the conditions of theorem 2 hold and that both X and its average $\overline{X}$ share the same equilibrium point. If the equilibrium point is exponentially stable for the averaged system, then the equilibrium point is exponentially stable for the original system.*

First-order averaging is not always sufficient to approximate the dynamics of a system. In these cases, second- or higher-order averaging techniques are needed.

Theorem 4 (second order averaging) *[1] Consider the mapping,*

$$y(t) = z(t) + \epsilon w(z(t), t)$$

and the initial value problem

$$\dot{z} = \epsilon \overline{X}(z) + \epsilon^2 \overline{Y}(z)$$

where,

$$w(x,t) = \int_0^t \left(X(x,\tau) - \overline{X}(x)\right) \mathrm{d}\tau + a(x)$$
$$Y(x,t) = DX(x,t) \cdot w(x,t) - Dw(x,t) \cdot \overline{X}(x),$$

such that $a(x)$ makes the time average of $w(x,t)$ vanish. Suppose that the following all hold: (1) $X(x,t)$ has a Lipschitz-continuous first derivative in x and is continuous on its domain of definition, and (2) $z(t)$ belongs to an interior subset of M on the time scale $\frac{1}{\epsilon}$. Then, $x(t) = y(t) + O(\epsilon^2)$, on the time scale $\frac{1}{\epsilon}$.

3 Series Expansions

In Section 4 we show that averaging theory is a natural consequence of nonlinear Floquet theory. To obtain n^{th}-order approximations, a series description of flows and vector fields will be required. The following review is a synopsis of relevant concepts from [6].

The general form for the equations of motion are

$$\frac{\mathrm{d}x}{\mathrm{d}t} = X(x,t), \quad x(t_0) = x_0, \tag{4}$$

where $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, X is in $C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ as a function of the x only, and is absolutely continuous as a function of t only. Sometimes we will write X_t for $X(\cdot, t)$. A solution to (4) is, $\mathcal{F}_{t_0,t} = \mathrm{Id} + \int_{t_0}^t X_\tau \cdot \mathcal{F}_{0,\tau} \mathrm{d}\tau$, which via successive substitutions results in the Volterra series

$$\mathcal{F}_{t_0,t} = \mathrm{Id} + \int_{t_0}^t X_\tau \mathrm{d}\tau + \int_{t_0}^t \mathrm{d}\tau_1 \int_{t_0}^{\tau_1} \mathrm{d}\tau_2 \, X_{\tau_1} \cdot X_{\tau_2} + \dots \tag{5}$$

Under appropriate conditions [6], the series $\mathcal{F}_{t_0,t}$ converges and represents the true system flow. In the autonomous case, the Volterra series reduces to an exponential

$$\mathcal{F}_{t_0,t} = \sum_{m=1}^{\infty} \frac{1}{m!} \left(\int_{t_0}^t X_\tau \mathrm{d}\tau\right)^m \equiv \exp\left(\int_{t_0}^t X_\tau \mathrm{d}\tau\right),$$

recovering the classical notion of an exponential for autonomous ordinary differential equations. Similarly, for the time-dependent case, the series is called the *chronological exponential in* X_τ[6]. It is written,

$$\mathcal{F}_{t_0,t} \cong \overrightarrow{\exp}\left(\int_{t_0}^t X_\tau \mathrm{d}\tau\right). \tag{6}$$

The asymptotic nature of this equality is discussed in [6]. The inverse to the chronological exponential is the *chronological logarithm* [6]; it depends on initial time, t_0,

$$\overrightarrow{\log}_{t_0} \overrightarrow{\exp}\left(\int_{t_0}^t X_\tau \, \mathrm{d}\tau\right) = X_t. \tag{7}$$

We will use the time-independent logarithm, simply called the *logarithm*, which is now reviewed.

Averaging theory seeks to find an autonomous vector field whose flow best represents the flow of the vector field in (4). Consider the existence of a vector field $\overrightarrow{\mathrm{V}}_{t_0,t}(X_\tau)$ such that the equality (in an asymptotic sense [6]) holds:

$$\overrightarrow{\exp} \int_{t_0}^t X_\tau \, \mathrm{d}\tau \cong \exp \overrightarrow{\mathrm{V}}_{t_0,t}(X_\tau). \tag{8}$$

A series expansion for this vector field, $\overrightarrow{\mathrm{V}}_{t_0,t}(X_\tau)$, exists, and assuming convergence [6], it is called the *logarithm*, i.e.,

$$\overrightarrow{\mathrm{V}}_{t_0,t}(X_\tau) \cong \ln \overrightarrow{\exp} \int_{t_0}^t Y_\tau \, \mathrm{d}\tau. \tag{9}$$

While the logarithm vector field, $\overrightarrow{\mathrm{V}}_{t_0,t}(X_\tau)$, depends on t, it is an autonomous vector field whose flow after unit time maps to the same point that is reached by the time-dependent flow at time t. Variation in the final time results in a new autonomous vector field.

The logarithm is at the heart of a series expansion approximation for the flow $\Phi^X_{t_0,t}$. The logarithm vector field is an infinite series of *variations* , $\overrightarrow{\mathrm{V}}^{(m)}_{t_0,t}(X_\tau)$

$$\overrightarrow{\mathrm{V}}_{t_0,t}(X_\tau) = \sum_{m=1}^{\infty} \overrightarrow{\mathrm{V}}^{(m)}_{t_0,t}(X_\tau) \tag{10}$$

where,

$$\overrightarrow{\mathrm{V}}^{(m)}_{t_0,t}(x_\tau) = \int_{t_0}^{t}\int_{t_0}^{\tau_1}\cdots\int_{t_0}^{\tau_{m-1}} \mathrm{v}^{(m)}(X_{\tau_1},\ldots,X_{\tau_m}) \, \mathrm{d}\tau_m\ldots\mathrm{d}\tau_2\mathrm{d}\tau_1$$

is the mth*-variation of the identity flow corresponding to the perturbation field* X_τ. The integrands, $\mathrm{v}^{(m)}(\cdot)$, of the variations are the sum of iterated Lie brackets, denoted here by ad, and thereby naturally reside within the space of vector fields. The first four integrand are given below:

$$\begin{aligned}
&\mathrm{v}^{(1)}(\xi_1) = \xi_1\\
&\mathrm{v}^{(2)}(\xi_1,\xi_2) = \frac{1}{2}\mathrm{ad}_{\xi_2}\xi_1\\
&\mathrm{v}^{(3)}(\xi_1,\xi_2,\xi_3) = \frac{1}{6}\left(\mathrm{ad}_{\xi_3}\mathrm{ad}_{\xi_2}\xi_1 + \mathrm{ad}_{\mathrm{ad}_{\xi_3}\xi_2}\xi_1\right)\\
&\mathrm{v}^{(4)}(\xi_1,\xi_2,\xi_3,\xi_4) = -\frac{1}{12}\left(\mathrm{ad}_{\mathrm{ad}_{\xi_4}\xi_3}\mathrm{ad}_{\xi_2}\xi_1\right.\\
&\qquad + \mathrm{ad}_{\mathrm{ad}_{\mathrm{ad}_{\xi_4}\xi_3}\xi_2}\xi_1 + \mathrm{ad}_{\xi_4}\mathrm{ad}_{\mathrm{ad}_{\xi_3}\xi_2}\xi_1\\
&\qquad \left. + \mathrm{ad}_{\xi_3}\mathrm{ad}_{\mathrm{ad}_{\xi_4}\xi_2}\xi_1\right).
\end{aligned} \tag{11}$$

By taking advantage of the Jacobi-Lie identity, the third and fourth variations were simplified to the form above. Under the condition that the first $(m-1)$ elements of the variations vanish, choosing only the first non-vanishing element to represent the flow gives,

Proposition 1 *[6] If at the point* $x \in \mathbb{R}^n$*, the following holds:* $\overrightarrow{V}^{(\alpha)}_{t_0,t}(X_\tau) = 0,\ \forall\alpha = 1,\ldots,m-1$*, then*

$$\overrightarrow{\exp}\int_{t_0}^{t} X_\tau\,\mathrm{d}\tau = x + \overrightarrow{V}^{(m)}_{t_0,t}(X_\tau) + O\left(\left(\int_{t_0}^{t}\|X_\tau\|_{s+2m}\,\mathrm{d}\tau\right)^{m+1}\right).$$

4 Averaging via Floquet Theory

This section reformulates averaging in terms of nonlinear Floquet theory, which provides a means to understand the flow of (4), rewritten here,

$$\dot{x} = X(x,t)\,, \quad x(0) = x_0, \tag{12}$$

with X, T-periodic, i.e., $X(x,t) = X(x,t+T)$.

Theorem 5 (Nonlinear Floquet Thm.) *Let* $\Phi^X_{0,t}$ *be the flow generated by the time-periodic differential equation (12). If the monodramy map,* $M = \Phi^X_{0,T}$*, has a logarithm, then the flow* $\Phi^X_{0,t}$ *can be represented as a composition of flows* $\Phi^X_{0,t} = P(t)\circ\exp(Yt)$*, where* P *is* T*-periodic, and*

$$Y = \frac{1}{T}\ln\overrightarrow{\exp}\left(\int_0^T X(x,\theta)\,\mathrm{d}\theta\right). \tag{13}$$

Proof: Our proof, while hinted at in [13], appears to be original. It follows its linear counterpart. If $\tau = t+T$, then both $\Phi^X_{0,t}$ and $\Phi^X_{0,\tau}$ are flows that differ by an invertible mapping, Ψ, with $\Phi^X_{0,t+T} = \Phi^X_{0,t}\circ\Psi$.

Assume, for now, that there exists an autonomous flow denoted by $\Phi^Y_{0,t}$ equaling Ψ at time T. Consider, $P(t) \equiv \Phi^X_{0,t}\circ\left(\Phi^Y_{0,t}\right)^{-1}$. The flow $P(t)$ is T-periodic.

$$\begin{aligned}
P(t+T) &= \Phi^X_{0,t+T}\circ\left(\Phi^Y_{0,t+T}\right)^{-1}\\
&= \Phi^X_{0,t}\circ\Phi^X_{0,T}\circ\left(\Phi^Y_{0,T}\right)^{-1}\circ\left(\Phi^Y_{0,t}\right)^{-1}\\
&= \Phi^X_{0,t}\circ\left(\Phi^Y_{0,t}\right)^{-1} = P(t).
\end{aligned}$$

The T-periodicity of the original vector field ensures, $\Phi^X_{0,t+T} = \Phi^X_{0,t}\circ\Phi^X_{0,T}$, implying $\Phi^Y_{0,T} = \Psi = \Phi^X_{0,T}$, i.e., Ψ is the monodramy map of the flow. Therefore, $\exp(YT) = \overrightarrow{\exp}\left(\int_0^T X_\tau\,\mathrm{d}\tau\right)$. Inverting via the logarithm yields equation (13), precisely the average of the T-periodic vector field; a connection that will be made more explicit later. ∎

Theorem 6 *If the monodramy map has a fixed point, then the actual flow has a periodic orbit whose stability is determined by the stability of the monodramy map.*

Proof: Stability can be found using a Poincaré map technique. From [17], the Poincaré map is the monodramy map. ∎

If the monodramy map is (exponentially) asymptotically stable, then the associated orbit is (exponentially) asymptotically stable.

Corollary 1 *If the flow of system (12) has a fixed point* x^**, as does the monodramy map, then stability of the actual flow may be determined using the monodramy map. In particular an (exponentially) asymptotically stable fixed point for the monodramy map implies an (exponentially) asymptotically stable fixed point for the actual system.*

Proof: Can be proven by appealing to the notions of D-stability and C-stability found in Pars [18]. ∎

These critical results link the stability of the averaged system to the stability of the original system. Explicit calculation of the monodramy map may be difficult; fortunately, the monodramy map is related to the autonomous vector field, Y. The flow of Y for time T gives the monodramy map.

Corollary 2 *[13] The stability properties of the logarithm of the monodramy map are equivalent to those of the monodramy map.*

Comment. In the context of linear systems, the above conclusions lead to the following well known fact for Floquet theory: calculation of the Floquet multipliers is equivalent to calculation of the Floquet exponents.

4.1 Application to Averaging Theory

Floquet theory is an important asset for averaging theory. Since the actual flow oscillates around the trajectory determined by the autonomous vector field Y, the monodramy map gives the turnpike behavior of the system [14]. To check stability, one must calculate the monodramy map or the full series expansion of its logarithm. Truncations of the series expansion of the logarithm may be sufficient, and can be thought of as partial averages of the system on the order of the truncation [13].

Let us reemphasize that averaging theory requires the parameter ϵ in the governing differential equations, (1), allowing for important proximity results and minimizing the error when truncating a series expansion.

The monodramy map is obtained from the autonomous vector field in Equation (13), which is a sum of (integrated) variations (10). The integrand of the m^{th} variation of the identity flow, $\mathrm{v}^{(m)}(\cdot)$, is given by the sum of Jacobi-Lie brackets (11). Since iterated Jacobi-Lie brackets are multi-linear, the parameter ϵ may be factored out. Therefore, Y will often be denoted by

$$Y = \sum_{\alpha=1}^{\infty} Y^{(\alpha)} \equiv \sum_{\alpha=1}^{\infty} \epsilon^{\alpha} \Lambda^{(\alpha)}. \tag{14}$$

Definition 1 *If the function* F *can be given by a series expansion, then* $\mathrm{Trunc}_m(\mathrm{F})$ *is a truncation of the series of the* $(m+1)$ *and higher terms.*

Definition 2 *A truncated series expansion is said to be a* stabilized expansion with respect to property P *if the inclusion of additional terms to the truncation does not affect property P of the expansion, i.e., if property* P *holds for all* $\mathrm{Trunc}_{m+k}(\mathrm{F})$, $k > 0$, *and* $\mathrm{Trunc}_m(\mathrm{F})$.

Definition 3 *[13] A* stabilized truncated series expansion with respect to linear stability *for vector field (14) is a truncated vector field series that has the same linear stability properties as the fully expanded vector field.*

These definitions imply that the eigenvalues for various truncations are calculated until they cannot be significantly affected by the ϵ^{m+1} vector field perturbation given by adding an extra term to the truncation. At this stabilized truncation, linear stability can be computed and used to determine the linear stability of the original system as per the previous propositions. See [13, 14] for particular examples involving matrix ODEs. For a control system, this is known as vibrational control.

If a truncation has not yet stabilized, it can still be capable of approximating the actual flow. The following theorem determines the interval of approximation validity. It is reminiscent of classical averaging theorems, only it holds for arbitrary truncations.

Theorem 7 *The* m^{th}*-order truncation of the logarithm of the monodramy map, denoted by* Y^{m}, *gives an* $(\mathrm{m}+1)^{th}$*-order approximation of the flow for finite time, i.e.,*

$$\exp(Yt) = \exp(Y^m t) + O\left(\epsilon^{m+1}\right) \tag{15}$$

on the time scale 1.

Proof: The difference in the flows can be understood by decomposing the total logarithm into a truncation and a truncated remainder, $Y = Y^m + \widetilde{Y}$, then using the variation of constants formula on the flow,

$$\Phi_{0,t}^{Y} = \Phi_{0,t}^{Z} \circ \Phi_{0,t}^{Y^m}, \text{ where } Z = \left(\Phi_{-t,0}^{Y^m}\right)^{*} \widetilde{Y}.$$

Thus the flow, $\Phi_{0,t}^{Z}$, acts as a perturbation to the final point of the flow of the truncated vector field. The size of this perturbation determines how far the truncated vector field flow is from the actual flow used to find the monodramy map. Since the pull-back is a linear operator, the vector field Z will scale with ϵ according to its contribution in $\widetilde{Y}$. By definition $\widetilde{Y}$ has a factor of $\epsilon^{(m+1)}$ that may be extracted. Therefore,

$$Z \equiv \epsilon^{m+1} \widehat{Z},$$

At this point, invoke Proposition 1 with $m = 0$.

$$\overrightarrow{\exp}\left(\int_{t_0}^{t} Z \,\mathrm{d}\tau\right) = \mathrm{Id} + O\left(\epsilon^{m+1} \int_{t_0}^{t} \left\|\widehat{Z}_{\tau}\right\|_{s} \mathrm{d}\tau\right)$$

By taking the maximum of $\widehat{Z}$ over space and time, one can arrive at,

$$\overrightarrow{\exp}\left(\int_{t_0}^{t} Z \,\mathrm{d}\tau\right) = \mathrm{Id} + O\left(\epsilon^{m+1} t^{m+2}\right). \tag{16}$$

The rest naturally follows on the time scale 1. ■

One thing to note is that the order of time may be increased at the cost of orders of approximation. For example, choosing the time scale $\epsilon^{-1/2}$ moves the order of approximation to $O\left(\epsilon^{(m+1)/(2m+4)}\right)$.

5 A General Averaging Theory

This section generalizes classical averaging theory to higher order. Nonlinear Floquet theory decomposes the flow of a time-periodic system into a time-periodic and autonomous flows. Classical higher-order averaging theory suggests the form of the autonomous vector field and the compensatory periodic flow. This section use series expansions and the chronological calculus to construct this flow and prove that it preserves the order of proximity of the truncated autonomous flow.

5.1 Truncations of the Floquet mapping

It was shown in the previous section that truncations of the autonomous Floquet flow provide approximations to the complete flow. Here it is likewise shown and argued that one may calculate truncations of the time-periodic mapping $P(t)$. Recall that $P(t)$ is given by

$$P(t) \equiv \Phi_{0,t}^{X} \circ \exp(-Yt). \tag{17}$$

Theorem 8 *An m^{th}-order truncation of the time-periodic Floquet mapping is of order $(m+1)$-close to the time-periodic Floquet mapping on the time scale* 1.

$$P(t) = \mathrm{Trunc}_m\left(P(t)\right) + O\left(\epsilon^{m+1}\right)$$

Proof: The proof is similar to that of Theorem 7. Consider the two flows $\Phi_{0,t}^X = \Phi^m + \widetilde{\Phi}$, and $\exp(-Yt) = \Psi^m + \widetilde{\Psi}$, where $\Phi^m = \mathrm{Trunc}_m\left(\Phi_{0,t}^X\right)$ and $\Psi^m = \mathrm{Trunc}_m\left(\exp(-Yt)\right)$. Then, $\Phi_{0,t}^X \circ \exp(-Yt) = \left(\Phi^m + \widetilde{\Phi}\right) \circ \left(\Psi^m + \widetilde{\Psi}\right)$. Expanding,

$$\Phi_{0,t}^X \circ \exp(-Yt) = \mathrm{Trunc}_m\left(\Phi^m \circ \Psi^m\right) + \widetilde{\Theta}.$$

The factor ϵ^{m+1} can be removed from $\widetilde{\Theta}$, i.e., $\widetilde{\Theta} \equiv \epsilon^{m+1}\widehat{\Theta}$, implying that,

$$\left|\left|\Phi_{0,t}^X \circ \exp(-Yt) - \mathrm{Trunc}_m\left(\Phi^m \circ \Psi^m\right)\right|\right|_s = \epsilon^{m+1}\left|\left|\widehat{\Theta}\right|\right|_s,$$

and the result follows on the time scale 1. ∎

It is not known a priori what properties the truncation of $P(t)$ will have, however one essential ingredient is time-periodicity. Therefore a realistic constraint to add to the truncation is time-periodicity, $P(t) = P(t+T)$. After extending the mapping to be periodic, it will be called the *amended truncation*.

Corollary 3 *If the amended truncation* $\mathrm{Trunc}_m\left(P(t)\right)$ *has period T, which is on the time scale 1, then the amended truncation is order $\epsilon^{(m+1)}$-close for all time.*[2]

Suppose that the following holds, $P(t) = \widetilde{P}(t)P_0$, with P_0 a time-independent transformation. It is possible to recover a different averaged vector field from this.

Proposition 2 *If the Floquet mapping has a time-independent bias, i.e., $P(t) = \widetilde{P}(t)P_0$. Then a new averaged vector field may be written $Z = (P_0)_* Y$.*

Proof: The autonomous flow is, $y(t) = \exp(Yt)y(0)$, and the actual flow is, $x(t) = P(t, y(t)) = \widetilde{P}(t)\circ P_0(y(t))$. Define the new variable, $z(t) = P_0 y(t)$, The evolution of $z(t)$ obeys the differential equation, $\dot{z} = (P_0)_* Y(z)$, and the solution becomes, $x(t) = \widetilde{P}(t) \circ \exp(Zt)$. ∎

Comment. A given initial condition might not be the actual average, so one should not expect the Floquet decomposition to result in the precise average. With the above construction, this problem may be overcome. The same is known to hold for linear Floquet theory.

The development of the vector fields and the compensatory flows required for an averaging theory is complete. An m^{th}-order averaged system is described by the Floquet mapping,

$$x(t) = \mathrm{Trunc}_{m-1}\left(P(y(t))\right) + O\left(\epsilon^m\right), \qquad (18)$$

[2]The corollary can be modified to get other orders of time at the sacrifice of orders of proximity.

and the autonomous vector field,

$$\dot{y} = \vec{\mathrm{V}}_{t_0,t}^m(X). \qquad (19)$$

5.2 First and Second-Order Averaging

As a simple application, let's revisit first order averaging. Unlike second order averaging, first order averaging does not involve a compensatory mapping. This is because,

$$\mathrm{Trunc}_0\left(P(t)\right) = \mathrm{Trunc}_0\left(\Phi_{0,t}^X \circ \exp(-Yt)\right) = \mathrm{Id}.$$

Since the compensatory mapping is the identity, the zero-mean and T-periodicity constraints are trivially satisfied. This leaves the autonomous vector field,

$$Y = \frac{1}{T}\int_0^T \epsilon X_\tau \,\mathrm{d}\tau = \epsilon\overline{X},$$

as the only important element in first-order averaging. Floquet theory can be applied to obtain the standard facts concerning stability of the average vector field flow and its relation to the actual flow.

Second order averaging. The benefits of the chronological calculus become more apparent when we reconstruct the 2^{nd}-order averaging theorem of Sanders and Verhulst. Truncating $P(t)$ results in

$$\mathrm{Trunc}_1\left(P(t)\right) = \mathrm{Id} + \epsilon\int_0^t \left(X_\tau - \overline{X}\right)\,\mathrm{d}\tau + O\left(\epsilon^2\right).$$

Sanders and Verhulst also require the flow's integral term to have a vanishing average [1]. This is a product of their proof technique, and relates to the comment after Proposition 2. Bogoliubov and Mitropolsky [2] obtain the same results. The autonomous vector field is,

$$Y = \epsilon\overline{X} + \frac{1}{2}\epsilon^2\overline{\left[\int_0^t X_\tau\,\mathrm{d}\tau\,, X_t\right]}.$$

The beauty of the series expansions approach lies in the fact that no new theorems are needed.

5.3 Higher-order Averaging

Here, we extend averaging to 3^{rd}- and 4^{th}-order.

Third order averaging. The truncation of $P(t)$ to second order is,

$$\begin{aligned}\mathrm{Trunc}_2\left(P(t)\right) &= \mathrm{Id} + \epsilon\int_0^t \left(X_\tau - \overline{X}\right)\,\mathrm{d}\tau \\ &+ \frac{1}{2}\epsilon^2\int_0^t \left(\left[\int_0^\tau X_s\,\mathrm{d}s\,, X_\tau\right] - \overline{\left[\int_0^t X_s\,\mathrm{d}s\,, X_t\right]}\right)\mathrm{d}\tau \\ &+ \frac{1}{2}\epsilon^2\int_0^t X_\tau\,\mathrm{d}\tau \circ \int_0^t X_\tau\,\mathrm{d}\tau - \epsilon^2\int_0^t X_\tau\,\mathrm{d}\circ\overline{X}\,t \\ &+ \frac{1}{2}\epsilon^2\overline{X}\circ\overline{X}\,t^2\end{aligned}$$

This truncation is not periodic, although $P(0) = P(T)$. To obtain a T-periodic function, define $P(\tau + kT) \equiv P(\tau)$, where $\tau \in [0, T)$ and $k \in \mathbb{Z}$. This must be done for higher-order truncations also. The averaged autonomous vector field is,

$$\begin{aligned}Y = \epsilon\overline{X} &+ \frac{1}{2}\epsilon^2\overline{\left[\int_0^t X_\tau\,\mathrm{d}\tau\,, X_t\right]} + \frac{1}{4}T\epsilon^3\left[\overline{X}, \overline{\left[\int_0^t X_\tau\,\mathrm{d}\tau\,, X_t\right]}\right] \\ &+ \frac{1}{3}\epsilon^3\overline{\left[\int_0^\tau X_{\tau_1}\,\mathrm{d}\tau_1\,, \left[\int_0^\tau X_{\tau_1}\,\mathrm{d}\tau_1\,, X_\tau\right]\right]}\end{aligned}$$

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$$
\begin{aligned}
\mathrm{Trunc}_3\left(P(t)\right) = \mathrm{Id} &+ \epsilon \int_0^t \left(X_\tau - \overline{X}\right) \mathrm{d}\tau \\
&+ \frac{1}{2}\epsilon^2 \int_0^t \left(\left[\int_0^\tau X_s \,\mathrm{d}s\,, X_\tau\right] - \overline{\left[\int_0^\sigma X_s \,\mathrm{d}s\,, X_\sigma\right]}\right) \mathrm{d}\tau \\
&+ \frac{1}{4}\epsilon^3 \left(\left[\int_0^t X_\tau \,\mathrm{d}\tau\,, \int_0^t \left[\int_0^\tau X_s \,\mathrm{d}s\,, X_\tau\right] \mathrm{d}\tau\right] - T\left[\overline{X}, \overline{\left[\int_0^t X_\tau \,\mathrm{d}\tau\,, X_t\right]}\right] t\right) \\
&+ \frac{1}{3}\epsilon^3 \int_0^t \left(\left[\int_0^\tau X_s \,\mathrm{d}s\,, \left[\int_0^\tau X_s \,\mathrm{d}s\,, X_\tau\right]\right] - \overline{\left[\int_0^t X_\tau \,\mathrm{d}\tau\,, \left[\int_0^t X_\tau \,\mathrm{d}\tau\,, X_t\right]\right]}\right) \mathrm{d}\tau \\
&+ \frac{1}{2}\epsilon^2 \int_0^t X_\tau \,\mathrm{d}\tau \circ \int_0^t X_\tau \,\mathrm{d}\tau - \epsilon^2 \int_0^t X_\tau \,\mathrm{d}\tau \circ \overline{X}\, t + \frac{1}{2}\epsilon^2 \overline{X} \circ \overline{X}\, t^2 \\
&+ \frac{1}{4}\epsilon^3 \int_0^t X_\tau \,\mathrm{d}\tau \circ \int_0^t \left[\int_0^\tau X_s \,\mathrm{d}s\,, X_\tau\right] \mathrm{d}\tau - \frac{1}{2}\epsilon^2 \int_0^t X_\tau \,\mathrm{d}\tau \circ \overline{\left[\int_0^t X_\tau \,\mathrm{d}\tau\,, X_t\right]}\, t \\
&- \frac{1}{2}\epsilon^3 \int_0^t \left[\int_0^\tau X_s \,\mathrm{d}s\,, X_\tau\right] \mathrm{d}\tau \circ \overline{X}\, t + \frac{1}{4}\epsilon^3 \overline{X} \circ \overline{\left[\int_0^t X_\tau \,\mathrm{d}\tau\,, X_t\right]}\, t^2 \\
&+ \frac{1}{4}\epsilon^3 \int_0^t \left[\int_0^\tau X_s \,\mathrm{d}s\,, X_\tau\right] \mathrm{d}\tau \circ \int_0^t X_\tau \,\mathrm{d}\tau + \frac{1}{4}\epsilon^3 \overline{\left[\int_0^t X_\tau \,\mathrm{d}\tau\,, X_t\right]} \circ \overline{X}\, t^2 \\
&+ \frac{1}{6}\epsilon^3 \left(\int_0^t X_\tau \,\mathrm{d}\tau \circ \int_0^t X_\tau \,\mathrm{d}\tau \circ \int_0^t X_\tau \,\mathrm{d}\tau\,, -\overline{X} \circ \overline{X} \circ \overline{X}\, t^3\right)
\end{aligned}
$$

Table 1: $\mathrm{Trunc}_3\left(P(t)\right))$

Fourth order averaging. The truncation of $P(t)$ to third order is found in table 1, and the averaged vector field in table 2.

A General Averaging Algorithm Although calculating higher-order averaging expansions is a tedious task, there is a simple algorithm for doing so (Table 3). The algorithm incorporates two dimensions of approximation. The first is the truncations of the logarithm vector field, $\mathrm{Trunc}_m\left(Y\right)$, which captures the system dynamics up to m^{th}-order. The second is the truncations of the compensation map, $\mathrm{Trunc}_{(m-1)}\left(P(t)\right)$, which gives m^{th}-order proximity between the two flows.

6 Conclusion

This paper generalized averaging theory to arbitrary order, giving both averaging formulas and theorems related to stability of the averaged system. While the framework can recover known averaging results, it is restricted to smooth or analytic vector fields. Further investigations into Volterra series expansions could relax this requirement.

A companion paper [3] uses this theory to construct algorithms for feedback control of underactuated driftless control systems. Because the averaged system demonstrates linear stability, one can appeal to the averaging stability theorems to conclude exponential stabilization of the control system. The algorithm also works to arbitrary orders of Jacobi-Lie brackets.

$$
\begin{aligned}
Y = \epsilon\overline{X} &+ \frac{1}{2}\epsilon^2 \overline{\left[\int_0^t X_\tau \,\mathrm{d}\tau\,, X_t\right]} + \frac{1}{4}T\epsilon^3 \left[\overline{X}, \overline{\left[\int_0^t X_\tau \,\mathrm{d}\tau\,, X_t\right]}\right] \\
&+ \frac{1}{3}\epsilon^3 \overline{\left[\int_0^\tau X_{\tau_1} \,\mathrm{d}\tau_1\,, \left[\int_0^\tau X_{\tau_1} \,\mathrm{d}\tau_1\,, X_\tau\right]\right]} \\
&+ \frac{1}{3}\epsilon^3 \overline{\left[\int_0^\tau X_{\tau_1} \,\mathrm{d}\tau_1\,, \left[\int_0^\tau X_{\tau_1} \,\mathrm{d}\tau_1\,, X_\tau\right]\right]} + \frac{1}{3}\epsilon^3 \left[a_1, a_{21}\right] \\
&+ \frac{1}{3}\epsilon^3 \left[a_1, \overline{\left[\int_0^\tau X_{\tau_1} \,\mathrm{d}\tau_1\,, X_\tau\right]}\right] + \frac{1}{3}\epsilon^3 \left[\overline{\int_0^\tau X_{\tau_1} \,\mathrm{d}\tau_1}\,, a_{21}\right] \\
&- \frac{1}{12}\epsilon^4 \overline{\int_0^\tau \left[\int_0^{\tau_1} \left[\int_0^{\tau_2} X_{\tau_3} \,\mathrm{d}\tau_3\,, X_{\tau_2}\right] \mathrm{d}\tau_2\,, \left[X_{\tau_1}, X_\tau\right]\right] \mathrm{d}\tau_1} \\
&- \frac{1}{12}\epsilon^4 \overline{\left[\int_0^\tau \left[\int_0^{\tau_1} \left[\int_0^{\tau_2} X_{\tau_3} \,\mathrm{d}\tau_3\,, X_{\tau_2}\right] \mathrm{d}\tau_2\,, x_{\tau_1}\right] \mathrm{d}\tau_1\,, X_\tau\right]} \\
&- \frac{1}{12}\epsilon^4 \overline{\int_0^\tau \left[\int_0^{\tau_1} X_{\tau_2} \,\mathrm{d}\tau_2\,, \left[\left[\int_0^{\tau_1} X_{\tau_2} \,\mathrm{d}\tau_2\,, X_{\tau_1}\right], X_\tau\right]\right] \mathrm{d}\tau_1}
\end{aligned}
$$

Table 2: $\mathrm{Trunc}_4\left(Y\right)$

1. Calculate the logarithm vector field, $\mathrm{Trunc}_m\left(Y\right)$.
2. Compute truncations of $\exp\left(-Yt\right)$ and $\overrightarrow{\exp}\left(\int_0^t X_\tau \mathrm{d}\tau\right)$.
3. Use the truncations for $\mathrm{Trunc}_{(m-1)}\left(P(t)\right)$.

Table 3: Algorithm for Computing the Average

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